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Space-time noncommutativity, discreteness of time and unitarity

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Abstract. Violation of unitarity for noncommutative field theory on compact space-times is considered. Although such theories are free of ultraviolet divergences they still violate unitarity, while in a usual field theory such a violation occurs when the theory is nonrenormalizable. The compactness of space-like coordinates implies discreteness of the time variable which leads to the appearance of unphysical modes and violation of unitarity even in the absence of a star-product in the interaction terms. Thus, this conclusion holds also for other quantum field theories with discrete time. Violation of causality, among others, occurs also in the case of the nonvanishing of the commutation relations between observables at space-like distances with a typical scale of noncommutativity. While this feature allows for a possible violation of the spin-statistics theorem, such a violation does not rescue the situation but makes causality violation scale as the inverse of the mass appearing in the considered model, i.e., it becomes even more severe. We also stress the role of smearing over the noncommutative coordinates entering the field operator symbols.

1 Introduction

It is generally believed that the picture of space-time as a manifold $\mathcal{M}$ should break down at very short distances, namely distances of the order of the Planck length. One possible approach to the description of physical phenomena at small distances is based on noncommutative (NC) geometry of the space-time. It has been shown that the noncommutative geometry naturally appears in string theory with a nonzero antisymmetric B-field [1]. Another approach, starting from the study of a relation between measurements at very small distances and black hole formations, has been developed in [2]. The essence of the noncommutative geometry consists in reformulating first the geometry in terms of commutative algebras of smooth functions, and then generalizing them to their noncommutative analogs in terms of operators (or, more generally, to use a C^*-algebra) generated by noncommuting space and time coordinates: $[\widehat{x}^\mu, \widehat{x}^\nu] \neq 0$.

The Hilbert (Fock) space for a commutative and the corresponding NC field theories are the same at the perturbative level. This is supported by the fact that the quadratic part of the action is not affected by a star-product. Moreover, this is the reason why there should be a map between any NC field theory and its commutative limit: the degrees of freedom are the same.

Noncommutative field theories with noncommutativity of only space coordinates (while the time remains a usual commutative variable) do not change crucially the standard quantum mechanical formalism (one can develop the usual Hamiltonian dynamics, define the corresponding Schrödinger picture, etc.). Of course, this kind of noncommutativity still essentially changes some properties of the theory: in particular, it becomes nonlocal in the space-like directions [3,4]. But such basic properties of physical models as causality and unitarity are satisfied. This can be traced back [3,4] to the fact that this theory describes low energy excitations of a D-brane in the presence of a background magnetic field (see [1] and references therein).

Field theories with space and *time* noncommutativity provide an interesting opportunity to test the possible breakdown of the conventional notion of time and the familiar framework of quantum mechanics at the Planck scale. As has been shown in [4–6], in the case of the model derived from string theory with a background electric field and in the flat space-time, noncommutativity of the time coordinates of the corresponding Minkowski space and the corresponding nonlocality in time result in violation of both the causality and unitarity conditions.

Thus, the question whether there exists some self-consistent theory with noncommutative time coordinate is of great interest. The analysis in [4–6] shows that the violation of the basic principles of causality and unitarity occurs at energies higher than the inverse scale of the pa-

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rameter of noncommutativity λ, i.e. for $E \gg \lambda^{-1}$. Thus, if some noncommutative theory implies an upper bound on possible values of the energy, one may hope that it is free of the problems with the violation of the basic physical principles. In [7], we have shown that space-time quantization on a two-dimensional cylinder leads to the energy spectrum, confined within the interval $E \in [0, \pi/\lambda]$. Therefore, it is natural to study the question of unitarity and causality for this case. It is worth noticing that this restriction on energy provides an improved ultraviolet behavior of the field theory on the NC cylinder: even planar diagrams in this case prove to be convergent (in contrast to the theory in the flat NC Minkowski space).

In fact, such a study has an even wider interest. The point is that the restriction on the energy values appears as a consequence of the discreteness of time (in the representation where the time coordinate operator is diagonal). On the other hand, attempts have been made to construct quantum field theories with discrete time which is considered to be not only an intermediate regularization (as in the lattice field theories) but to have fundamental physical meaning [8] (for recent attempts see, e.g., the series of papers [9] and references therein). The problem of unitarity has not been investigated for this kind of models.

In this letter, we shall show that the situation with the violation of the unitarity condition on the cylinder is even more severe than that in the case of the flat space-time. More precisely, due to the discreteness of the time evolution, the unitarity requirement is violated even by planar diagrams (which do not carry a trace of the star-product). That means that the result is valid for any theory with a discrete time variable and not only for the field theory with the space-time noncommutativity.

This letter is arranged as follows. In Sect. 2, we present some facts about the noncommutative cylinder and the corresponding Φ^4-field theory, necessary for further study. In Sect. 3, we prove the violation of unitarity for planar diagrams in one-loop approximation. Section 4 is devoted to conclusions and remarks.

2 Field theory on a noncommutative cylinder

The points on a commutative cylinder C can be specified by a real parameter $t \in \mathbb{R}$ and two complex parameters $x_\pm = \rho \mathrm{e}^{\pm \mathrm{i}\alpha}$. The fields possess the following expansion:

$$\Phi(t,\alpha) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \tilde{\Phi}_k(\omega) \mathrm{e}^{\mathrm{i}k\alpha - \mathrm{i}\omega t}. \qquad (1)$$

In the noncommutative case [7] the parameters $t, x_\pm$ are replaced by operators $\hat{t}, \hat{x}_\pm$ satisfying the commutation relations

$$[\hat{t}, \hat{x}_\pm] = \pm\lambda \hat{x}_\pm, \quad [\hat{x}_+, \hat{x}_-] = 0, \qquad (2)$$

and the same constraint equation as in the commutative case: $\hat{x}_+\hat{x}_- = \rho^2$. The dimensionful (with the dimension of length) parameter λ is an analog of the tensor θ in the case of the Heisenberg-like commutation relation in the flat Minkowski space. However, in the present case, the actual parameter of the noncommutativity is the dimensionless parameter $\eta = \lambda/\rho$.

The operators $\hat{t}, \hat{x}_\pm$ can be realized in the auxiliary Hilbert space $\mathcal{H} = L^2(S^1, \mathrm{d}\alpha)$ as follows:

$$\hat{t} = -\mathrm{i}\lambda\partial_\alpha, \quad \hat{x}_\pm = \rho \mathrm{e}^{\pm \mathrm{i}k\alpha}. \qquad (3)$$

We specify the self-adjoint extension of ∂_α by postulating its system of eigenfunctions: $\partial_\alpha f_k(\alpha) = \mathrm{i}k f_k(\alpha)$, $f_k(\alpha) = \mathrm{e}^{\mathrm{i}k\alpha}$, $k \in \mathbb{Z}$. Thus, we are dealing with a unitary irreducible representation of the two-dimensional Euclidean group $E(2)$ specified by the value of the Casimir operator $\hat{x}_+\hat{x}_- = \rho^2$.

In analogy with the commutative case, we take the fields to be operators in $\mathcal{H} = L^2(S^1, \mathrm{d}\alpha)$ possessing the operator Fourier expansion:

$$\Phi(\hat{t},\hat{\alpha}) = \sum_{k=-\infty}^{\infty} \int_{-\pi/\lambda}^{+\pi/\lambda} \frac{\mathrm{d}\omega}{2\pi} \tilde{\Phi}_k(\omega) \mathrm{e}^{\mathrm{i}k\hat{\alpha} - \mathrm{i}\omega\hat{t}}. \qquad (4)$$

For simplicity, we shall consider a real scalar field theory which corresponds to the condition $\Phi^\dagger(\hat{t},\hat{\alpha}) = \Phi(\hat{t},\hat{\alpha})$. It is important that since the spectrum of $\hat{t}$ is discrete, we have $t = \lambda n$, $n \in \mathbb{Z}$, and the integration over $\mathrm{d}\omega$ goes only over a finite interval $(-\pi/\lambda, +\pi/\lambda)$. We point out that the operator Fourier expansion (4) is invertible:

$$\tilde{\Phi}_k(\omega) = \frac{1}{2\pi} \mathrm{Tr}\,[\mathrm{e}^{-\mathrm{i}k\hat{\alpha}+\mathrm{i}\omega\hat{t}} \Phi(\hat{t},\hat{\alpha})]. \qquad (5)$$

This follows straightforwardly from the formula

$$\frac{1}{2\pi} \mathrm{Tr}\,[\mathrm{e}^{-\mathrm{i}k'\hat{\alpha}+\mathrm{i}\omega'\hat{t}} \mathrm{e}^{\mathrm{i}k\hat{\alpha}-\mathrm{i}\omega\hat{t}}] = \delta_{k'k}\delta^{(S)}(\lambda\omega' - \lambda\omega), \qquad (6)$$

where $\delta^{(S)}(\varphi)$ denotes the δ-function on a circle. The inverse *usual* Fourier transform of $\tilde{\Phi}_k(\omega)$ yields an analog of the Weyl symbol $\Phi(n\lambda, \alpha)$ on the cylinder:

$$\Phi(n\lambda,\alpha) = \sum_{k=-\infty}^{\infty} \int_{-\pi/\lambda}^{+\pi/\lambda} \frac{\mathrm{d}\omega}{2\pi} \tilde{\Phi}_k(\omega) \mathrm{e}^{\mathrm{i}k\alpha - \mathrm{i}\lambda\omega n}. \qquad (7)$$

Notice that since $\Phi(n\lambda, \alpha)$ is not a function on the whole commutative cylinder, but takes values only at discrete points of the time variable, this is not the canonical Weyl symbol. The latter can be constructed if one considers all possible self-adjoint extensions of the operator ∂_α on a circle. Since this is not important for our consideration, we drop further discussion of this possibility.

The star-product for the fields $\Phi(n\lambda, \alpha)$ has a form which is very close to that appearing in flat space-time:

$$\begin{aligned} &\Phi_1(n\lambda,\alpha) \star \Phi_2(n\lambda,\alpha) = \\ &\quad \mathrm{e}^{(\mathrm{i}\lambda/2)((\partial/\partial t_1)(\partial/\partial\varphi_2) - (\partial/\partial t_2)(\partial/\partial\varphi_1))} \\ &\quad \times\Phi_1(n\lambda + t_1, \alpha + \varphi_1) \\ &\quad \times\Phi_2(n\lambda + t_2, \alpha + \varphi_2)\big|_{\substack{t_1=t_2=0 \\ \varphi_1=\varphi_2=0}}, \end{aligned} \qquad (8)$$

where $t_1, t_2, \varphi_1, \varphi_2$ are auxiliary continuous variables.

On the commutative cylinder, the d'Alembertian can be expressed through the Poisson brackets [7]:

$$\Box\Phi = \{t, \{t, \Phi\}\} + \rho^{-2}\{x_+, \{x_-, \Phi\}\}, \tag{9}$$

where

$$\{F, G\} = \frac{\partial F}{\partial \varphi}\frac{\partial G}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial G}{\partial \varphi}.$$

We generalize this to the noncommutative case by replacing the Poisson brackets by commutators: $\{.,.\} \to (1/(\mathrm{i}\lambda))[.,.]$. This gives the free action on the noncommutative cylinder in the form

$$\begin{aligned} S_0^{(\mathrm{NC})}[\hat{\Phi}] &= \pi\eta \mathrm{Tr}\left\{-\frac{1}{\lambda^2}[\widehat{x}_+, \hat{\Phi}][\widehat{x}_-, \hat{\Phi}] + \frac{1}{\lambda^2}[\widehat{t}, \hat{\Phi}]^2 - \mu^2\hat{\Phi}^2\right\} \\ &= \frac{\eta}{2}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} \mathrm{d}\alpha \left[(\delta\Phi(n,\alpha))^2 - \left(\frac{\partial\Phi(n,\alpha)}{\partial\alpha}\right)^2 - \mu^2\right]. \end{aligned} \tag{10}$$

Here

$$\delta\Phi(n,\alpha) = \frac{1}{\eta}[\Phi(n+1,\alpha) - \Phi(n,\alpha)]$$

(we have simplified the notation for the field: $\Phi(n\lambda,\alpha) \to \Phi(n,\alpha)$), and μ is the dimensionless parameter related to the mass: $\mu = \rho m$. As usual for the Weyl symbol, the star-product disappears from the trace for a product of any two operators. In the case of a field theory in a flat space, this leads to the free action which formally looks like the one on commutative space. In the case of the cylinder, we have the trace of noncommutativity even in the free action: it reveals itself in discrete time derivatives. We stress that this is an intrinsic property of field theories on noncommutative manifolds with *compact* space-like dimensions and appears in any formalism and for any operator symbols.

The Φ^4-interaction term contains, in general, the star-product:

$$\begin{aligned} S_{\mathrm{int}}^{(\mathrm{NC})} &= \frac{g}{4!}2\pi \mathrm{Tr}\{\Phi^4(\widehat{t}, \hat{\alpha})\} \\ &= \frac{g\eta}{4!}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{d}\alpha(\Phi(n,\alpha)\star\Phi(n,\alpha))^2. \end{aligned} \tag{11}$$

In the momentum representation, this star-product results in the appearance of the factors $\cos[\lambda(\omega k' - \omega' k)]$ (here ω, ω', k, k' are the energies and momenta entering the vertex). These factors grow both in the upper and lower half-planes of the complex-energy plane and prevent us from the use of the standard Cutkosky cutting rules and, eventually, lead to the violation of unitarity. Although in the case of the cylinder, we have to consider only a strip $\mathcal{R}e\,\omega \in [-\pi/\lambda, \pi/\lambda]$ in the complex-energy plane, the consideration proves to be essentially the same as in the case of flat space-time [5] and we do not repeat it.

For a possible attempt to rescue the theory, one may try to modify the interaction term. One possibility is to define the action through a specific ordering prescription for the noncommuting operators $\widehat{t}$ and $\hat{\alpha}$ (a situation not obtainable from the known string theories). In particular, a $t\alpha$-"normal" ordering (i.e., the requirement that in the operator expression for the action all operators $\widehat{t}$ be posed to the left of all operators $\hat{\alpha}$) leads to the disappearance of the star-product in the interaction term [7]:

$$S_{\mathrm{int}}^{(\mathrm{NC},t\alpha)} = \frac{g\eta}{4!}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{d}\alpha\Phi^4(n,\alpha). \tag{12}$$

In a flat space-time, such a version of noncommutative field theory exactly coincides with the usual commutative QFT (except that now one deals with operator symbols instead of the usual fields, so that the interpretation of events in space and time requires additional smearing, while all calculations and results in the momentum space remain the same as in the usual QFT). On the contrary, in the case of the cylinder, even after the ordering, we still have the trace of the noncommutativity, namely, the discreteness of the time variable. Thus, it is interesting to verify (see next section) whether such a variant of the noncommutative field theory preserves unitarity. Another motivation for this study is the persistent attempts to construct quantum field theories with improved ultraviolet behavior starting from the postulate of the discreteness of time [9].

3 Unitarity in theories with discrete time

The free field equation of motion derived from the action (10) reads as follows:

$$(\bar{\delta}\delta - \partial_\alpha\partial_\alpha + \mu^2)\Phi(n,\alpha) = 0 \tag{13}$$

(here $\bar{\delta}f(n) \equiv [f(n) - f(n-1)]/\eta$), and the corresponding propagator has the form

$$D_0^{(\mathrm{NC})}(\omega, k) = \frac{1}{\Omega^2(\omega) - k^2/\rho^2 - m^2 + \mathrm{i}\varepsilon}, \tag{14}$$

where

$$\Omega = \frac{2}{\lambda}\sin\left(\frac{\lambda\omega}{2}\right). \tag{15}$$

The modes which satisfy the condition $k^2 \le \Lambda^2 \equiv 4/\eta^2 - \mu^2$ correspond to the usual oscillating solutions of (13) and resemble the solutions in the continuous-time physics. On the contrary, the modes with $k^2 > \Lambda^2 \equiv 4/\eta^2 - \mu^2$ correspond to solutions growing or decreasing in time and these, as we shall show soon, are unphysical. Correspondingly, the propagator has two types of singular points:

(1) the oscillating modes with $k^2 \le \Lambda^2$ produce poles in the complex-energy plane at $\pm\omega_k \mp \mathrm{i}\varepsilon$, where $\omega_k > 0$ is defined by the equality

$$\sin^2\left(\frac{\lambda\omega_k}{2}\right) = \frac{\eta^2}{4}(k^2 + \mu^2);$$

(2) the modes with $k^2 > \Lambda^2$ produce poles at $\omega_k = \pi/\lambda \pm \mathrm{i}S_k$, where $S_k > 0$ is defined by the equality

$$\cosh(\lambda S_k/2) = \frac{\eta^2}{2}(k^2+\mu^2) - 1. \tag{16}$$

In order to realize the physical meaning of the two types of modes, we use the method of the transfer matrix (see, e.g. [10]). The transfer matrix T_k for a given mode $\Phi_k(n) = (2\pi)^{-1}\int \mathrm{d}\alpha \Phi(n,\alpha)\exp\{-\mathrm{i}k\alpha\}$ in the discrete time field theory under consideration has the form

$$T_k = \exp\left\{\mathrm{i}\left[\frac{(\Phi_k(n+1)-\Phi_k(n))^2}{\eta} - \frac{\eta}{2}(k^2+\mu^2)\left(\Phi_k^2(n+1)+\Phi_k^2(n)\right)\right]\right\}.$$

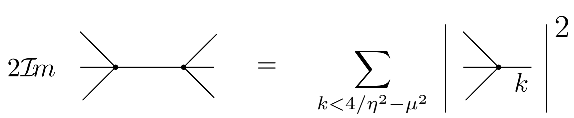
Then the calculation of the corresponding Hamiltonian, defined by $\widehat{H} = -\mathrm{i}/\lambda \ln T$, shows that while for the oscillating modes we obtain a harmonic oscillator Hamiltonian with the frequency W defined by the relation $\sin(W\eta/2) = \eta(k^2+\mu^2)^{1/2}/2$, the modes with $k^2 > 4/\eta^2 - \mu^2$ correspond to a Hamiltonian which is not a positive definite (bounded from below) operator. Thus, these modes are unphysical ones and we have to study unitarity within the subspace of the oscillating modes. In other words, we have to check that the unphysical states decouple from the physical ones similarly to the ghost fields in gauge field theory or to unstable states [11].

We shall check the unitarity condition, i.e.

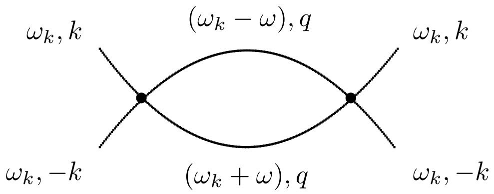
$$2\mathcal{I}m\, M_{ab} = \sum_c M_{ac}M_{cb}, \tag{17}$$

for the on-shell transition matrix elements M_{ab} between states a and b in second order of the perturbation theory for the interaction of the form (12) (i.e., for planar diagrams in the case of the standard noncommutative field theory or for a theory with the $t\alpha$-ordering defined above, or for a theory which simply starts from postulating discreteness of time).

One can easily check that at the tree level the unitarity condition in the physical sector is indeed satisfied:



Next, we consider the s-channel 1-loop Feynman diagram



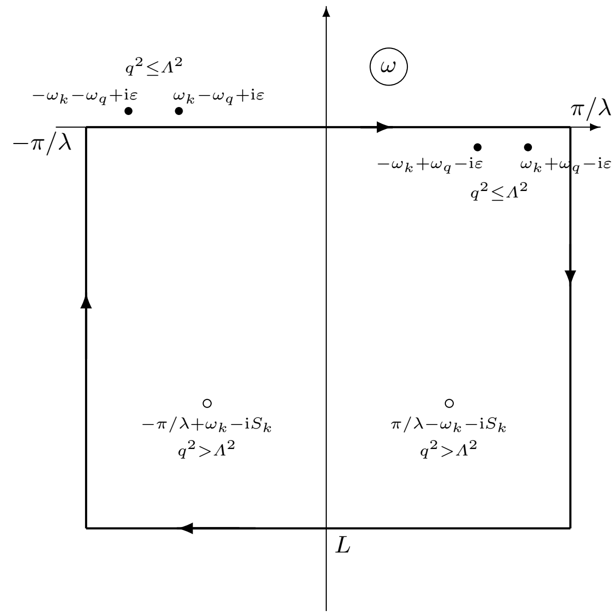


Fig. 1. The singularities for the two type of modes and the contour of integration for the calculation of the imaginary part of the amplitude; $\omega_q = (2/\lambda)\arcsin[(\eta/2)(q^2+\mu^2)^{1/2}]$

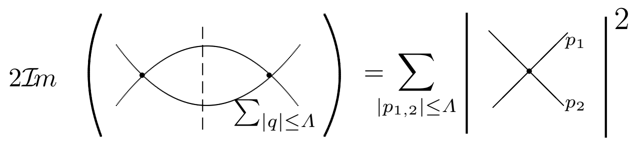
in the center-of-mass frame (one can easily check that the corresponding t- and u-channel diagrams have no branch cut singularities above the threshold). The corresponding contribution to the matrix element reads

$$\mathrm{i}M = \frac{g^2}{2}\sum_{q=-\infty}^{\infty}\int_{-\pi/\lambda}^{\pi/\lambda} \mathrm{d}\omega D_0^{(\mathrm{NC})}(\omega_k+\omega, q) D_0^{(\mathrm{NC})}(\omega_k-\omega, q). \tag{18}$$

The calculation of the imaginary part of the amplitude can be carried out by closing the contour of integration in the complex-energy plane downward as is shown in Fig. 1.

In Fig. 1, the filled circles denote the usual Feynman-like poles at the points $\omega = \pm\omega_k + (2/\lambda)\arcsin[(\eta/2)(k^2+\mu^2)^{1/2}] - \mathrm{i}\varepsilon$ (in the lower half-plane) and at $\omega = \pm\omega_k - (2/\lambda)\arcsin[(\eta/2)(k^2+\mu^2)^{1/2}] + \mathrm{i}\varepsilon$ (in the upper half-plane), appearing for the oscillating modes with $q^2 \le \Lambda^2$. The small empty circles denote the positions of the singularities for the unphysical modes with $q^2 > \Lambda^2$ at $\omega = (\pm\pi/\lambda \mp \omega_k) - \mathrm{i}S_k$. Here $S_k > 0$ is the solution of (16) (there exist symmetrical singularities in the upper half-plane, but these are not important for us). The closing of the contour is possible due to the facts that the contributions from its vertical parts cancel each other due to the periodicity in the energy variable, while the lower horizontal part gives a vanishing contribution when the distance L to the real axis goes to infinity. The latter is true only for the interaction vertex *without* the star-product cosine factors (i.e. for planar diagrams, or for theories with the $t\alpha$-"normal" ordering defined above or simply for a theory with discrete time).

We separate the sum in (18) into two parts: $\sum_{q\in\mathbb{Z}} = \sum_{|q|\le\Lambda} + \sum_{|q|>\Lambda}$ and, first, we consider the part with the oscillation modes $|q| \le \Lambda$. Then, proceeding in the usual way [12] (see also, e.g., [13] and references therein) and taking the residues of the corresponding poles, one can show that *this* part of the sum already gives the contribution which saturates the unitarity condition in the physical sector of the oscillating modes:



Unfortunately, the part of the sum corresponding to the unphysical modes also gives a contribution to the imaginary part of the amplitudes due to the poles indicated in Fig. 1 by the empty circles. In general, this non-zero contribution looks rather cumbersome, but for the particular value of the external energy, namely for $\omega_k = \pi/(2\lambda)$, it becomes quite simple:

$$[2\mathcal{I}m\, M]^{(\mathrm{unph})}|_{\omega_k=\pi/(2\lambda)} = \frac{g^2}{4(2\pi)^2} \sum_{|q|>\Lambda} (q^2+\mu^2)^{3/2}(q^2+\mu^2-4/\eta^2)^{-1/2}.$$

The proof of the unitarity violation for theories with flat space-like dimensions (e.g., for the one proposed in [9]) goes essentially in the same way (with the only distinction that the sums over momentum modes is substituted by the corresponding integrals). Thus, the theories with a discrete time variable do not satisfy the unitarity condition.

4 Conclusions and remarks

We have shown that the transition to noncommutative spaces with compact space dimensions does not help in restoring unitarity in the theories with space and *time* noncommutativity. We also have proved the more general statement that any theory with discrete time variable meets with the same problem.

It is clear from the previous section that in the absence of the cosine factors in vertices, coming from the star-product, the origin of the nonunitarity in theories with discrete time is the appearance of the unphysical nonoscillating modes. Notice that if one takes a specific value of the parameter of noncommutativity, namely $\eta = 2\pi/N$ (N is a positive integer), the basic operator exponentials in (4) satisfy the commutation relation

$$\mathrm{e}^{\mathrm{i}\hat{t}}\mathrm{e}^{\mathrm{i}\hat{\alpha}} = \mathrm{e}^{\mathrm{i}2\pi/N}\mathrm{e}^{\mathrm{i}\hat{\alpha}}\mathrm{e}^{\mathrm{i}\hat{t}},$$

and possess finite-dimensional representations [14]. This implies that for a small mass $m \sim \mathcal{O}\left(N^{-2}\right)$ appearing in (10), we can get rid of the unphysical modes. However, the choice of the noncommutativity parameter as indicated above, means, actually, a transition to the quantum torus [15], i.e., to a manifold with closed (compact) time-like curves. As is well known [16], theories on such manifolds, even in the commutative case, have their own problems with causality and the formulation of the unitarity condition. Therefore, we do not pursue this possibility further here.

Theories with space-time noncommutativity suffer also from the violation of causality. In [4], this fact was demonstrated by the example of the scattering of wave-packets. Another possibility to see the violation of (micro)causality is to calculate the matrix elements of equal-time commutators of some observables in this theory. We note that in physical applications one has finally to smear over the noncommutative coordinates in field operator symbols, since the symbols themselves do not reflect the values of the operator coordinates [7]: $\overline{\Phi(x)} \equiv \langle x|\,\hat{\Phi}(\widehat{x})\,|x\rangle$, where $|x\rangle$ is, for instance, a (maximally localized) coherent state. This smearing would make a difference in the interpretation of violation of, e.g., (micro)causality if the violation would be occurring only at the scales of the order of λ and not growing with the energy.

In particular, for quadratic observables we have[1]:

$$\langle 0|\,[\overline{\Phi^2(x)}, \overline{\Phi^2(y)}\,|\mathbf{p},\mathbf{k}\rangle\,|_{x^0=y^0} \approx \mathrm{e}^{-(\mathbf{x}-\mathbf{y})^2/(4\lambda^2)}. \qquad (19)$$

The asymptotic behavior (19) has been derived for the case when the distance between two points is large: $|\mathbf{x}-\mathbf{y}| \gg \lambda$, and the momenta $\mathbf{p},\mathbf{k}$ are not too high (the result looks similar for both cases of a flat space-time and the cylinder, if the distance is understood accordingly). For large values of momenta $\mathbf{p}$ and $\mathbf{k}$ of a two-particle state $|\mathbf{p},\mathbf{k}\rangle$, however, the exponential damping (19) does not occur anymore. Actually, this violation of the causality (as well as that observed in [4]) can be interpreted as the impossibility of a precise simultaneous measurement of space and time coordinates, in accordance with the original idea presented in [2]. We also mention that in NC theory with the $t\alpha$-"normal"("time-space") ordering prescription all the commutators between observables would vanish at space-like distances.

Another interesting question concerning the NC field theories is the problem of causality and the spin-statistics theorem [17] (see also, e.g., [18]). As is well-known, in the usual commutative quantum field theory the requirement of vanishing of commutators for physical observables at space-like distances (i.e., causality) leads uniquely to the spin-statistics theorem. Since in NC field theory such commutation relations are not equal to zero as explained above, one has, in principle, no longer the same arguments for the derivation of the spin-statistics relation and thus the modification of the latter is not excluded[2]. We have studied several most natural modifications of the usual spin-statistics (i.e., modifications of the commutation relations for creation and annihilation operators) and found

[1] Notice that all the vacuum expectation values (vacuum–vacuum matrix elements) of the commutators between observables at space-like distances identically vanish for the NC field theories exactly like in the commutative case

[2] It is interesting that the CPT-theorem remains valid in NC field theories [19], but it is known that the CPT-theorem requires weaker assumptions than the spin-statistics one does

out that they are not only unable to help in the restoration of causality (cf. (19)), but instead they lead to commutation relations which are nonvanishing as in (19) but with a scale of the mass of the field m instead of $1/\lambda$ as in (19), which is even a more severe violation of causality. This violation of (micro)causality is of exactly the same form as which occurs in the usual commutative field theories when one modifies the spin-statistics relation.

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